hep-th/0511007

Accelerating black diholes and static black di-rings

Edward Teo

The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

Department of Physics, National University of Singapore, Singapore 119260

Abstract

We show how a recently discovered black ring solution with a rotating 2-sphere can be turned into two new solutions of Einstein–Maxwell–dilaton theory. The first is a four-dimensional solution describing a pair of oppositely charged, extremal black holes—known as a black dihole—undergoing uniform acceleration. The second is a five-dimensional solution describing a pair of concentric, static extremal black rings carrying opposite dipole charges—a so-called black di-ring. The properties of both solutions, which turn out to be formally very similar, are analyzed in detail. We also present, in an appendix, an accelerating version of the Zipoy–Voorhees solution in four-dimensional Einstein gravity.

1. Introduction

The complexity of the Einstein field equations, both in four and especially in higher dimensions, means that exact solutions describing physically interesting space-times are not easy to come by. Instead of solving the field equations directly, one of the more fruitful ways of obtaining new solutions has been to generate them starting from known solutions. There is, by now, a large catalogue of 'tricks' developed to do precisely this, although it is not always clear *a priori* that one can obtain physically meaningful solutions by a given solution-generating technique.

One of the most spectacular examples of a physically interesting solution obtained in this way in recent years, is the five-dimensional rotating black ring solution of Emparan and Reall [1]. This black ring is so-called because it has an event horizon of topology $S^1 \times S^2$, and it is the first and only known example of an asymptotically flat, regular vacuum solution with a horizon of non-spherical topology. It rotates in the S^1 ring direction, creating the necessary centrifugal force to balance its gravitational self-attraction. This solution was obtained using the direct relationship between five-dimensional Einstein gravity and its four-dimensional Kaluza–Klein reduction. The starting point was the previously known four-dimensional electrically charged C-metric solution in Kaluza–Klein theory, describing a Kaluza–Klein black hole undergoing uniform acceleration. Emparan and Reall lifted this solution to five dimensions with appropriate analytic continuations, and then showed that it can be reinterpreted as a rotating black ring.

In hindsight, it is possible to understand why there should at all be a relationship between four-dimensional accelerating black holes and five-dimensional black rings. This is most clearly seen from the so-called rod structures of both space-times [2]. The rod structure of an accelerating space-time would necessarily contain a semi-infinite rod for the time coordinate, corresponding to the acceleration horizon. Such a rod, when analytically continued to a space-like coordinate, has the different interpretation as an axis of rotation. It turns out that the procedure employed by Emparan and Reall has the effect of turning the acceleration horizon into a rotational axis for the new fifth coordinate (while leaving the rest of the rod structure essentially unchanged). This results in the four-dimensional black hole turning into a five-dimensional black ring around the new axis.

This relationship between the two different space-times has a few interesting implications. For example, consider taking the limit of the black ring in which the S^1 ring radius is sent to infinity. Note that increasing the ring radius corresponds, in the four-dimensional picture, to increasing the distance between the black hole and the acceleration horizon. The

limit in which the acceleration horizon is pushed to infinity is, in fact, equivalent to taking the zero-acceleration limit of the Kaluza–Klein C-metric. Thus, we see that there is a formal equivalence between taking the infinite-ring radius limit in five dimensions, and taking the zero-acceleration limit in four dimensions.

Recently, a rotating black ring different from the Emparan–Reall black ring was discovered by Figueras [3].* The difference lies in the fact that the rotation is now in the azimuthal direction of the S^2, rather than in the S^1 direction. Because of this, the self-attraction of the ring is not balanced by any centrifugal force, and there are necessarily conical singularities inside the ring to prevent it from collapsing. For the record, this black ring was discovered by "educated guesswork" [3], from the limits it was expected to reproduce.

In the light of the above-described relationship between five-dimensional black rings and four-dimensional accelerating black holes, it is natural to wonder what this new black ring solution would correspond to in four dimensions. An important clue comes from taking the infinite-radius limit of it, in which one obtains a Kerr black hole extended along the (straight) fifth direction. Upon taking the appropriate analytic continuations interchanging the time and fifth coordinates, this becomes the Euclideanized version of the Kerr black hole with a flat time direction. The Kaluza–Klein reduction of this space-time would therefore be the zero-acceleration limit of the, as yet unidentified, four-dimensional solution.

Now, it turns out that the Kerr black hole has indeed been used before, in precisely such a fashion, to generate a new solution of Kaluza–Klein theory [5]. The resulting solution was found to describe a static, magnetic dipole source. This dipole source can be interpreted as two oppositely charged, extremal Kaluza–Klein black holes in static equilibrium, a configuration known as a black dihole [6]. Thus, we may conclude that the Figueras black ring can be turned, by appropriate analytic continuations and Kaluza–Klein reduction, into a new four-dimensional solution describing an *accelerating* black dihole.

We shall see in Sec. 2 that this interpretation is indeed correct. In fact, we would be able to generalize this Kaluza–Klein solution to one of Einstein–Maxwell–dilaton theory with arbitrary exponential dilaton coupling. The properties of the accelerating dihole solution are then studied in detail. We first show how various limits of it can be taken, to obtain other known solutions such as the extremal dilatonic C-metric. It is then shown that there are necessarily conical singularities along the axis of symmetry, which are responsible for the acceleration of the black holes, as well as for keeping them apart. This solution is also

*This solution was also claimed to have been discovered in [4], although in different and indeed much more complicated coordinates.

generalized to include a background magnetic field, although it is found that this magnetic field is unable to fully replace the roles of the conical singularities.

There is another solution that can be generated from the Figueras black ring, by applying a higher-dimensional analogue of the procedure used in [5]. Specifically, the black ring is Euclideanized (with $t \to ix^6$), and a flat time direction is added. The resulting six-dimensional vacuum solution can then be dimensionally reduced along x^6 to give a new solution of five-dimensional Kaluza–Klein theory. Recall that when this procedure is applied to the five-dimensional Myers–Perry black hole, a static, extremal magnetic black ring results [7, 8]. Since diametrically opposite points of this ring carry opposite charges, the ring has a zero net charge; for this reason, it is also known as a dipole ring. Now, it turns out that this new solution describes a pair of concentric, extremal dipole rings. Furthermore, they carry opposite charges, in the sense that points on either ring with the same S^1 coordinate have opposite charges. In analogy with the case of black diholes, we shall call such a double-ring configuration a 'black di-ring'.

This solution is the subject of Sec. 3. In fact, we would be able to generalize the Kaluza–Klein di-ring to one of five-dimensional Einstein–Maxwell–dilaton theory with arbitrary exponential dilaton coupling.[†] It turns out that this di-ring solution is formally very similar to the accelerating dihole solution, and so the analysis of its properties would proceed analogously. Various limits of it are first taken to confirm its interpretation. It is then shown that conical singularities are necessarily present in the system to counterbalance the self-attraction of the inner ring, as well as the mutual attraction of the two rings. This solution is also generalized to include a background magnetic field, although it is found that this magnetic field is not able to fully balance the forces present and remove all the conical singularities from the system.

The paper concludes with some brief remarks, including possible applications to string theory. There is also an appendix, in which an accelerating version of the Zipoy–Voorhees solution [10, 11] in four-dimensional vacuum Einstein gravity is presented. This solution was deduced from the fact that a special case of the Zipoy–Voorhees solution (the so-called Darmois solution) arises as the coincident limit of the black dihole solution in pure Einstein–Maxwell theory.

[†]Rings in this theory have previously been considered, e.g., in [9], but these have a *net* electric charge, and so are different from the ones considered in this paper.

2. Accelerating black dihole solution

Our starting point is the metric for the black ring with a rotating 2-sphere, as found by Figueras [3]:

$$\begin{aligned}
\mathrm{d}s^2 = & -\frac{H(y,x)}{H(x,y)}\left[\mathrm{d}t - \frac{2maAy(1-x^2)}{H(y,x)}\,\mathrm{d}\varphi\right]^2 \\
& + \frac{1}{A^2(x-y)^2}\,H(x,y)\Bigg[-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} - \frac{(1-y^2)F(x)}{H(x,y)}\,\mathrm{d}\psi^2 \\
& \qquad + \frac{\mathrm{d}x^2}{(1-x^2)F(x)} + \frac{(1-x^2)F(y)}{H(y,x)}\,\mathrm{d}\varphi^2\Bigg], \qquad (2.1)
\end{aligned}$$

where

$$F(\xi) = 1 + 2mA\xi + (aA\xi)^2, \qquad H(\xi_1,\xi_2) = 1 + 2mA\xi_1 + (aA\xi_1\xi_2)^2, \qquad (2.2)$$

and m, a and A are positive constants. The angular coordinate ψ parametrizes the S^1 direction of the ring, while (x,φ) parametrize the rotating S^2. Further details on this solution may be found in [3].

We now perform the analytic continuation $t \to ix^5$, $\psi \to it$, $a \to ia$, and dimensionally reduce along the fifth direction x^5. The result is a solution to Kaluza–Klein theory with the following action:

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\mathrm{e}^{-\sqrt{3}\phi}F^2\right), \qquad (2.3)$$

where $F_{ab} \equiv \partial_a A_b - \partial_b A_a$. The four-dimensional metric is

$$\begin{aligned}
\mathrm{d}s^2 = & \frac{1}{A^2(x-y)^2}\Bigg[\left(\frac{H(y,x)}{H(x,y)}\right)^{\frac{1}{2}}(1-y^2)F(x)\,\mathrm{d}t^2 + \left(\frac{H(y,x)}{H(x,y)}\right)^{-\frac{1}{2}}(1-x^2)F(y)\,\mathrm{d}\varphi^2 \\
& + \frac{(H(x,y)H(y,x))^{\frac{1}{2}}}{K_0^2}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} + \frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right)\Bigg], \qquad (2.4a)
\end{aligned}$$

while the gauge potential and dilaton are, respectively,

$$A_\varphi = \frac{2maAy(1-x^2)}{H(y,x)}, \qquad (2.4b)$$

$$\phi = -\frac{\sqrt{3}}{2}\ln\left(\frac{H(y,x)}{H(x,y)}\right), \qquad (2.4c)$$

where, now,

$$F(\xi) = 1 + 2mA\xi - (aA\xi)^2, \qquad H(\xi_1,\xi_2) = 1 + 2mA\xi_1 - (aA\xi_1\xi_2)^2. \qquad (2.5)$$

Note that we have introduced a constant K_0 into (2.4a), whose significance would be discussed below. This solution is manifestly static and axisymmetric about the φ-axis. As can be seen from (2.4b), it carries a pure magnetic charge.

Kaluza–Klein theory is, in fact, a special case of a more general class of Einstein–Maxwell–dilaton theories with the action

$$\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left(R-\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\mathrm{e}^{-\alpha\phi}F^2\right), \tag{2.6}$$

where α is a non-negative parameter known as the dilaton coupling. It would prove to be convenient to introduce the new parameter N given by

$$N=\frac{4}{1+\alpha^2}\,,\qquad 0<N\leq 4\,. \tag{2.7}$$

The action (2.3) is recovered when $N=1$, while other integer values of N also correspond to well-known cases. One would naturally like to keep the dilaton coupling general as far as possible.

Fortunately, there is a systematic procedure known [12, 13] to turn a static, axisymmetric, magnetic solution of (2.3) into one of (2.6), valid for general dilaton coupling. When applied to (2.4), the resulting solution is

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{A^2(x-y)^2}\Bigg[\left(\frac{H(y,x)}{H(x,y)}\right)^{\frac{N}{2}}(1-y^2)F(x)\,\mathrm{d}t^2+\left(\frac{H(y,x)}{H(x,y)}\right)^{-\frac{N}{2}}(1-x^2)F(y)\,\mathrm{d}\varphi^2\\
&\qquad +\frac{(H(x,y)H(y,x))^{\frac{N}{2}}}{K_0^2\,G(x,y)^{N-1}}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)}+\frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right)\Bigg], &\text{(2.8a)}\\
A_\varphi &= \frac{2\sqrt{N}maAy(1-x^2)}{H(y,x)}\,, &\text{(2.8b)}\\
\phi &= -\frac{\alpha N}{2}\ln\left(\frac{H(y,x)}{H(x,y)}\right), &\text{(2.8c)}
\end{aligned}$$

where $F(\xi)$ and $H(\xi_1,\xi_2)$ are as in (2.5), and the new function $G(x,y)$ is given by

$$G(x,y)=(1+mA(x+y)-a^2A^2xy)^2-(m^2+a^2)A^2(1-xy)^2. \tag{2.9}$$

It is to this general solution that we shall instead direct our attention. We will see that it describes a black dihole undergoing uniform acceleration, and the analysis of its properties will closely follow both that of the C-metric describing a single accelerating black hole (see, e.g., [14, 15]), and the non-accelerating dihole solution [6].

Let us write the two roots of $F(y)$ as $-\frac{1}{r_\pm A}$, where

$$r_\pm \equiv m \pm \sqrt{m^2+a^2}\,. \tag{2.10}$$

Furthermore, we assume that $0 < r_+A < 1$. Then to ensure that the metric (2.8a) has the correct space-time signature, the (x, y) coordinates have to take the range

$$-1 \leq x \leq 1\,, \qquad -\frac{1}{r_+A} \leq y \leq -1\,. \tag{2.11}$$

The curvature invariants of the metric show that asymptotic infinity is at $x = y = -1$, while the only curvature singularities in this range lie at the two points $y = -\frac{1}{r_+A}$, $x = \pm 1$.

We now construct the rod structure [2] of this metric, which would help to reveal its physical significance. Note that the metric component g_{tt} vanishes at the two points $y = -\frac{1}{r_+A}$, $x = \pm 1$, which indicates that these are the locations of two extremal black holes.* This is consistent with the above-mentioned fact that the curvature is infinite at the same two points. g_{tt} also vanishes along the semi-infinite line $y = -1$, which may be identified as an acceleration horizon [2]. Thus, we arrive at the rod structure in Fig. 1, which clearly corresponds to two extremal black holes undergoing acceleration. The part of the symmetry axis between the two black holes is given by $y = -\frac{1}{r_+A}$; that between the first black hole (as labelled in Fig. 1) and the acceleration horizon is $x = 1$, while that joining the second black hole to infinity is $x = -1$.

This physical interpretation of the solution can be confirmed by taking various limits of it. A familiar one is the zero-acceleration limit, in which the acceleration horizon is effectively pushed to infinity. This is achieved by performing the coordinate transformation

$$t = A\tilde{t}\,, \qquad x = \cos\theta\,, \qquad y = -\frac{1}{rA}\,, \tag{2.12}$$

and then taking $A \to 0$. In this limit, the solution (2.8) reduces to

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(\frac{\triangle + a^2\sin^2\theta}{\Sigma}\right)^{\frac{N}{2}} \left[-\mathrm{d}\tilde{t}^2 + \frac{\Sigma^N}{K_0^2\,(\triangle + (m^2+a^2)\sin^2\theta)^{N-1}}\left(\frac{\mathrm{d}r^2}{\triangle} + \mathrm{d}\theta^2\right)\right] \\ & + \left(\frac{\triangle + a^2\sin^2\theta}{\Sigma}\right)^{-\frac{N}{2}} \triangle\sin^2\theta\,\mathrm{d}\varphi^2, \end{aligned} \tag{2.13a}$$

*Recall that extremal black holes have the universal property that their corresponding rods have zero length (see, e.g., [13]).

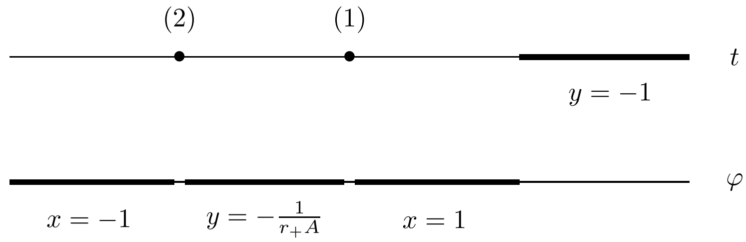

Figure 1: The rod structure of the accelerating dihole solution.

$$A_\varphi = -\frac{2\sqrt{N} mra \sin^2\theta}{\triangle + a^2 \sin^2\theta}\,, \tag{2.13b}$$

$$\phi = -\frac{\alpha N}{2} \ln\left(\frac{\triangle + a^2\sin^2\theta}{\Sigma}\right), \tag{2.13c}$$

where

$$\triangle = r^2 - 2mr - a^2, \qquad \Sigma = r^2 - a^2\cos^2\theta\,. \tag{2.14}$$

This is but the solution for a dilatonic black dihole [16, 17]. It not only confirms the presence of two extremal magnetic black holes in the rod structure of Fig. 1, it also proves that they have opposite charges. Furthermore, it is known from the dihole solution that the parameters m and a are related to the mass/charge of the black holes and the separation between them, respectively, and this would continue to be true in the general solution.

Another limit one could consider is to push the second black hole of Fig. 1 to infinity, while leaving the first black hole behind. One would then expect to obtain the usual C-metric describing a single extremal black hole undergoing uniform acceleration. This limit is achieved if we set $A = \tilde{A}/(1 + a\tilde{A})$, perform the coordinate transformation

$$x = 1 - \frac{(1-\tilde{x})(1+m\tilde{A}\tilde{y})}{a\tilde{A}(\tilde{x}-\tilde{y})}\,, \qquad y = -1 + \frac{(1+\tilde{y})(1+m\tilde{A}\tilde{x})}{a\tilde{A}(\tilde{x}-\tilde{y})}\,, \tag{2.15}$$

and then take $a \to \infty$. The solution (2.8) becomes, for an appropriate choice of K_0,

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\tilde{A}^2(\tilde{x}-\tilde{y})^2}\Bigg[&\left(\frac{F(\tilde{y})}{F(\tilde{x})}\right)^{\frac{N}{2}} (1-\tilde{y}^2)F(\tilde{x})^2\,\mathrm{d}t^2 + \left(\frac{F(\tilde{y})}{F(\tilde{x})}\right)^{-\frac{N}{2}} (1-\tilde{x}^2)F(\tilde{y})^2\,\mathrm{d}\varphi^2 \\ &+ (F(\tilde{x})F(\tilde{y}))^{\frac{4-N}{2}} \left(-\frac{\mathrm{d}\tilde{y}^2}{(1-\tilde{y}^2)F(\tilde{y})^2} + \frac{\mathrm{d}\tilde{x}^2}{(1-\tilde{x}^2)F(\tilde{x})^2}\right)\Bigg], \end{aligned} \tag{2.16a}$$

$$A_\varphi = -\sqrt{N} m (1-\tilde{x})\,, \tag{2.16b}$$

$$\phi = -\frac{\alpha N}{2} \ln\left(\frac{F(\tilde{y})}{F(\tilde{x})}\right), \tag{2.16c}$$

where $F(\tilde{\xi}) = 1 + m\tilde{A}\tilde{\xi}$. Indeed, it can be seen that this is just the dilatonic generalization of the extremal magnetic C-metric [14], in the form written in [15].

It is also possible to zoom in to each of the black holes, without the need to send the other black hole or the acceleration horizon to infinity. To zoom in to the first black hole, we perform the coordinate transformation

$$t = \frac{A\tilde{t}}{\sqrt{(1 - r_+ A)(1 + r_- A)}}\,, \tag{2.17a}$$

$$x = 1 - \frac{1 + r_+ A}{2\sqrt{m^2 + a^2}}\sqrt{\frac{1 + r_- A}{1 - r_+ A}}\, r(1 - \cos\theta)\,, \tag{2.17b}$$

$$y = -\frac{1}{r_+ A}\left[1 - \frac{1 + r_+ A}{2r_+}\sqrt{\frac{1 - r_+ A}{1 + r_- A}}\, r(1 + \cos\theta)\right], \tag{2.17c}$$

and then take the new coordinate r to be smaller than any other length scale present. For an appropriately chosen K_0, the solution (2.8) becomes

$$\begin{aligned} \mathrm{d}s^2 &= g(\theta)^{\frac{N}{2}}\left[-\left(\frac{r}{Q}\right)^{\frac{N}{2}}\mathrm{d}\tilde{t}^2 + \left(\frac{r}{Q}\right)^{-\frac{N}{2}}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)\right] \\ &\qquad + \left(\frac{r}{Q}\,g(\theta)\right)^{-\frac{N}{2}} r^2\sin^2\theta\,\mathrm{d}\varphi^2, \end{aligned} \tag{2.18a}$$

$$A_\varphi = -\frac{aQ\sqrt{N}}{\sqrt{m^2 + a^2}}\sqrt{\frac{1 + r_- A}{1 - r_+ A}}\,\frac{1 - \cos\theta}{g(\theta)}\,, \tag{2.18b}$$

$$\phi = -\frac{\alpha N}{2}\ln\left(\frac{r}{Q}\,g(\theta)\right), \tag{2.18c}$$

where we have defined

$$Q = \frac{mr_+}{\sqrt{m^2 + a^2}}\sqrt{\frac{1 + r_- A}{1 - r_+ A}}\,, \tag{2.19a}$$

$$g(\theta) = \frac{1}{2}\left[1 + \cos\theta + \frac{a^2}{m^2 + a^2}\,\frac{1 + r_- A}{1 - r_+ A}\,(1 - \cos\theta)\right]. \tag{2.19b}$$

Note that if the factor $g(\theta)$ were replaced by 1, the metric (2.18a) would be precisely that near the horizon of an extremal dilaton black hole (with the horizon located at $r = 0$). In actual fact, this horizon is distorted away from spherical symmetry by a non-trivial $g(\theta)$, and this may be attributed to the effect of the other black hole as well as the acceleration. Furthermore, a calculation of the curvature invariants reveals that this horizon is regular only in the non-dilatonic case $N = 4$. This is consistent with the fact that extremal dilaton black holes have horizons which are actually null singularities.

Similarly, we may zoom in to the second black hole with the coordinate transformation

$$t = \frac{A\tilde{t}}{\sqrt{(1+r_+A)(1-r_-A)}}\,, \tag{2.20a}$$

$$x = -1 + \frac{1-r_+A}{2\sqrt{m^2+a^2}}\sqrt{\frac{1-r_-A}{1+r_+A}}\,r(1-\cos\theta)\,, \tag{2.20b}$$

$$y = -\frac{1}{r_+A}\left[1 - \frac{1-r_+A}{2r_+}\sqrt{\frac{1+r_+A}{1-r_-A}}\,r(1+\cos\theta)\right]. \tag{2.20c}$$

The resulting solution is again given by (2.18) and (2.19), but with the replacement $A \to -A$. This reflects the asymmetry of the two black holes with respect to the location of the acceleration horizon; in other words, they are affected differently by the acceleration. But the above remarks for the first black hole still apply.

This distortion of the two black-hole horizons is related to the fact that there are conical singularities along the symmetry axis attached to the black holes. If we take the angle φ to have the usual periodicity 2π, then the deficit angles along the three different parts of the axis are

$$\delta_{(x=1)} = 2\pi\left[1 - (1+2mA-a^2A^2)^{\frac{N}{2}}\,K_0\right], \tag{2.21a}$$

$$\delta_{(x=-1)} = 2\pi\left[1 - (1-2mA-a^2A^2)^{\frac{N}{2}}\,K_0\right], \tag{2.21b}$$

$$\delta_{\left(y=-\frac{1}{r_+A}\right)} = 2\pi\left\{1 - \left[\left(1+\frac{m^2}{a^2}\right)(1-r_+^2A^2)\right]^{\frac{N}{2}}\,K_0\right\}. \tag{2.21c}$$

In general, it is possible to remove only one of the three conical singularities with an appropriate choice of the constant K_0. For example, if we remove the conical singularity along $x=1$ with the choice $K_0 = (1+2mA-a^2A^2)^{-\frac{N}{2}}$, then there is a positive deficit angle along $x=-1$. This can be interpreted as a semi-infinite cosmic string pulling on the dihole pair. Alternatively, we can remove the conical singularity along $x=-1$ with the choice $K_0 = (1-2mA-a^2A^2)^{-\frac{N}{2}}$, resulting in a *negative* deficit angle along $x=1$. This can be interpreted as a strut pushing on the dihole pair. The strut actually continues past the acceleration horizon and joins up with a 'mirror' dihole pair on the other side of it, although a change of coordinates is needed to see this [18].

In both the preceding cases, there is also in general a conical singularity along the axis $y=-\frac{1}{r_+A}$ between the pair of black holes. When $K_0 = (1-2mA-a^2A^2)^{-\frac{N}{2}}$, it can be checked that the deficit angle along $y=-\frac{1}{r_+A}$ is always negative. Thus, in addition to the strut along $x=1$ pushing on the first black hole, there is another strut between the two

black holes keeping them apart. On the other hand, when $K_0 = (1 + 2mA - a^2A^2)^{-\frac{N}{2}}$, the deficit angle along $y = -\frac{1}{r_+A}$ can take either sign. A particularly interesting situation is when it vanishes, which occurs when

$$1 + \frac{m^2}{a^2} = \frac{1 + r_-A}{1 - r_+A} \,. \tag{2.22}$$

In this situation, the *only* conical singularity in the space-time is the cosmic string along $x = -1$ pulling on the second black hole. The first black hole is accelerated along in the same direction by virtue of the attraction between the two black holes. Note that in this case, the distortion factor (2.19b) for the first black hole's horizon is identically one; since there are no conical singularities attached to this black hole, its horizon is perfectly spherically symmetric.

Now, recall that it is possible to immerse the dilatonic C-metric [14] or dihole solution [6] in a background magnetic field, by means of a Harrison-type transformation [14]. By adjusting its field strength appropriately, such a magnetic field could replace the role of the conical singularities in accelerating the black holes or keeping them apart. It is similarly possible to immerse the accelerating dihole solution (2.8) in a background magnetic field. The resulting solution is

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{A^2(x-y)^2}\Bigg[\left(\frac{H(y,x)\Lambda}{H(x,y)}\right)^{\frac{N}{2}}(1-y^2)F(x)\,\mathrm{d}t^2 + \left(\frac{H(y,x)\Lambda}{H(x,y)}\right)^{-\frac{N}{2}}(1-x^2)F(y)\,\mathrm{d}\varphi^2 \\
&\qquad + \frac{(H(x,y)H(y,x)\Lambda)^{\frac{N}{2}}}{K_0^2\,G(x,y)^{N-1}}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} + \frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right)\Bigg], && \text{(2.23a)} \\
A_\varphi &= \frac{2\sqrt{N}maAy(1-x^2)}{H(y,x)\Lambda} + \frac{B(1-x^2)}{H(y,x)^2\Lambda}\left((2maAy)^2(1-x^2) + \frac{F(y)H(x,y)^2}{A^2(x-y)^2}\right), && \text{(2.23b)} \\
\phi &= -\frac{\alpha N}{2}\ln\left(\frac{H(y,x)\Lambda}{H(x,y)}\right), && \text{(2.23c)}
\end{aligned}$$

where the function Λ is given by

$$\Lambda = \left(1 + \frac{2B}{\sqrt{N}}\,\frac{maAy(1-x^2)}{H(y,x)}\right)^2 + \frac{B^2}{N}\,\frac{(1-x^2)F(y)}{A^2(x-y)^2}\left(\frac{H(x,y)}{H(y,x)}\right)^2, \tag{2.24}$$

and B is a new parameter governing the strength of the background magnetic field.

The properties of this solution can be analyzed in the usual manner. For example, we may zoom in to the near-horizon region of either black hole using the same transformations

as above. For the first black hole, we again obtain the solution (2.18) and (2.19), except that A_φ acquires the extra factor $1-\frac{2}{\sqrt{N}}\frac{Bmr_+}{a}$, and $g(\theta)$ is replaced by

$$g(\theta)=\frac{1}{2}\left[1+\cos\theta+\frac{a^2}{m^2+a^2}\,\frac{1+r_-A}{1-r_+A}\left(1-\frac{2}{\sqrt{N}}\,\frac{Bmr_+}{a}\right)^2(1-\cos\theta)\right]. \tag{2.25}$$

The near-horizon solution for the second black hole is obtained from this by replacing $A\to -A$. Thus, the magnetic field does contribute to the distortion of the black-hole horizons.

One may also check for the presence of conical singularities in this space-time in the usual manner. It turns out that the deficit angles along the $x=\pm1$ axes are unchanged from (2.21a,b). Thus, the above arguments are still valid: there must either be a cosmic string along $x=-1$ pulling on the dihole pair, or a strut along $x=1$ pushing on it. The background magnetic field cannot be the source of acceleration, unlike in the case of the charged C-metric. The reason for this is simple: the dihole pair has a zero net charge, and so its overall motion is unaffected by the magnetic field. On the other hand, the conical deficit along $y=-\frac{1}{r_+A}$ is now replaced by

$$\delta_{\left(y=-\frac{1}{r_+A}\right)}=2\pi\left\{1-\left[\left(1+\frac{m^2}{a^2}\right)(1-r_+^2A^2)\left(1-\frac{2}{\sqrt{N}}\,\frac{Bmr_+}{a}\right)^{-2}\right]^{\frac{N}{2}}K_0\right\}. \tag{2.26}$$

This means that the magnetic field can play a role in balancing the forces between the two black holes. In fact, by adjusting B appropriately, it is always possible to remove the conical singularity along $y=-\frac{1}{r_+A}$, regardless of the choice of K_0. Again, if all the conical singularities attached to a particular black hole is removed, it can be checked that the horizon of this black hole becomes perfectly spherically symmetric.

Finally, we note that electric versions of the above solutions may be obtained by performing the duality transformation:

$$\phi'=-\phi\,,\qquad F'_{ab}=\frac{\mathrm{e}^{-\alpha\phi}}{2}\,\varepsilon_{ab}{}^{cd}F_{cd}\,. \tag{2.27}$$

Applying this transformation to the solution (2.23) gives the following electric gauge potential:

$$\begin{aligned}A'_t=&\ \frac{2\sqrt{N}maAx(1-y^2)}{H(x,y)}\left(1-\frac{B}{\sqrt{N}}\frac{ay(1-x^2)}{x-y}\right)^2\\&-\frac{B}{A^2(x-y)^2}\Big((1-2xy+x^2)\Big(1+4mAy\Big(1+\frac{Ba}{\sqrt{N}}\Big)\Big)\\&\qquad -Ax(1-y^2)\Big(2m+a^2Ax+\frac{2B}{\sqrt{N}}ma(1+x^2)\Big)\Big)\,.\end{aligned} \tag{2.28}$$

This dual solution describes an electrically charged dihole accelerating in a background electric field. Its properties are otherwise very similar to those of the magnetic solution.

3. Black di-ring solution

Our starting point is again the black ring metric (2.1). We now perform the analytic continuation $t \to ix^6$, $a \to ia$, to obtain a Euclidean version of this metric. A flat time direction is then added, resulting in a six-dimensional vacuum Einstein metric with Lorentzian signature. When dimensionally reduced along x^6, we obtain a solution to five-dimensional Kaluza–Klein theory with the action

$$\frac{1}{16\pi G}\int \mathrm{d}^5x\,\sqrt{-g}\left(R-\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\mathrm{e}^{-2\sqrt{\frac{2}{3}}\phi}F^2\right). \tag{3.1}$$

Explicitly, the solution is

$$\begin{aligned}
\mathrm{d}s^2 &= -\left(\frac{H(y,x)}{H(x,y)}\right)^{\frac{1}{3}}\mathrm{d}t^2+\frac{1}{A^2(x-y)^2}\left(H(y,x)H(x,y)^2\right)^{\frac{1}{3}}\Bigg[-\frac{(1-y^2)F(x)}{H(x,y)}\,\mathrm{d}\psi^2 \\
&\qquad +\frac{1}{K_0^2}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)}+\frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right)+\frac{(1-x^2)F(y)}{H(y,x)}\,\mathrm{d}\varphi^2\Bigg], && (3.2\text{a})\\
A_\varphi &= \frac{2maAy(1-x^2)}{H(y,x)}, && (3.2\text{b})\\
\phi &= -\sqrt{\frac{2}{3}}\ln\left(\frac{H(y,x)}{H(x,y)}\right), && (3.2\text{c})
\end{aligned}$$

where $F(\xi)$ and $H(\xi_1,\xi_2)$ are as in (2.5). Note that we have introduced a constant K_0 into (3.2a), which will be adjusted appropriately below.

As in four dimensions, the action (3.1) is but a special case of a more general class of five-dimensional Einstein–Maxwell–dilaton theories with the action

$$\frac{1}{16\pi G}\int \mathrm{d}^5x\,\sqrt{-g}\left(R-\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\mathrm{e}^{-\alpha\phi}F^2\right). \tag{3.3}$$

It will turn out to be convenient to introduce the new parameter

$$N=\frac{12}{4+3\alpha^2}\,,\qquad 0<N\le 3\,, \tag{3.4}$$

so that the Kaluza–Klein case corresponds to $N=1$. Again, it is straightforward to generalize the Kaluza–Klein solution (3.2) to other values of the dilaton coupling. The general

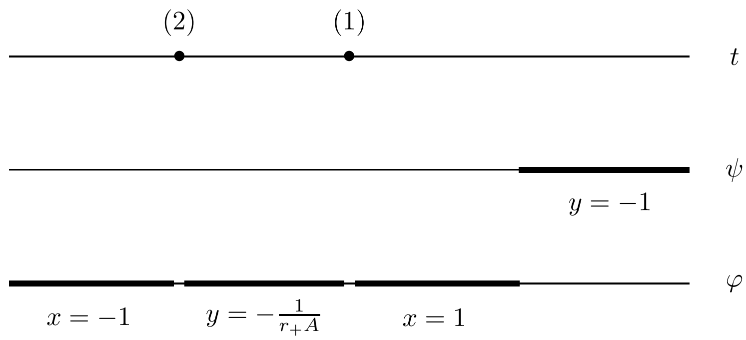

Figure 2: The rod structure of the di-ring solution.

solution is

$$\begin{aligned} \mathrm{d}s^2 &= -\left(\frac{H(y,x)}{H(x,y)}\right)^{\frac{N}{3}} \mathrm{d}t^2 + \frac{1}{A^2(x-y)^2} \left(H(y,x)H(x,y)^2\right)^{\frac{N}{3}} \left[-\frac{(1-y^2)F(x)}{H(x,y)^N}\,\mathrm{d}\psi^2 \right. \\ &\left. + \frac{1}{K_0^2\, G(x,y)^{N-1}} \left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} + \frac{\mathrm{d}x^2}{(1-x^2)F(x)} \right) + \frac{(1-x^2)F(y)}{H(y,x)^N}\,\mathrm{d}\varphi^2 \right], \quad (3.5a) \\ A_\varphi &= \frac{2\sqrt{N} maAy(1-x^2)}{H(y,x)}\,, \quad (3.5b) \\ \phi &= -\frac{\alpha N}{2} \ln\left(\frac{H(y,x)}{H(x,y)}\right), \quad (3.5c) \end{aligned}$$

where the function $G(x,y)$ is the same as in (2.9). Given the formal similarity of this solution with the four-dimensional accelerating dihole (2.8)—in fact, only the metric differs—its physical interpretation and properties would not be too difficult to deduce. In particular, the (x,y) coordinates continue to take the range (2.11).

The rod structure for this space-time is depicted in Fig. 2. Indeed, the resemblance with that in Fig. 1 is obvious, the only difference being that the acceleration horizon of the four-dimensional space-time is now the semi-infinite axis of the new angular coordinate ψ. Hence the space-time contains two concentric, static extremal black rings circling around the ψ-axis. This interpretation can be confirmed by taking various limits of the solution.

The simplest limit to take is that of infinite ring radius, which as explained in the introduction, is formally identical to taking the zero-acceleration limit in four dimensions. In this case, the coordinate transformation is given by (compare (2.12)):

$$\psi = A\tilde{\psi}\,, \qquad x = \cos\theta\,, \qquad y = -\frac{1}{rA}\,. \quad (3.6)$$

Upon taking the limit $A \to 0$, we obtain the metric

$$\begin{aligned} \mathrm{d}s^2 = &\left(\frac{\triangle + a^2 \sin^2\theta}{\Sigma}\right)^{\frac{N}{3}} \left[-\mathrm{d}t^2 + \mathrm{d}\tilde{\psi}^2 + \frac{\Sigma^N}{K_0^2\,(\triangle + (m^2+a^2)\sin^2\theta)^{N-1}}\left(\frac{\mathrm{d}r^2}{\triangle} + \mathrm{d}\theta^2\right)\right] \\ &+ \left(\frac{\triangle + a^2 \sin^2\theta}{\Sigma}\right)^{-\frac{2N}{3}} \triangle \sin^2\theta\, \mathrm{d}\varphi^2, \end{aligned} \tag{3.7}$$

where $\triangle$ and Σ are as in (2.14). The associated gauge field and dilaton are given by (2.13b) and (2.13c), respectively; of course, this is with the understanding that N is now defined by (3.4) rather than (2.7). This solution is in fact a five-dimensional generalization of the black dihole solution, whereby the two oppositely charged, extremal magnetic black holes are now black strings extended in the $\tilde{\psi}$ direction. These black strings carry a so-called 'local charge' [19], defined by

$$\mathcal{Q} = \frac{1}{4\pi}\int_{S^2} F\,, \tag{3.8}$$

where the S^2 encloses a point along the string. The two strings have opposite local charges, in the sense that when the S^2 encloses points on both strings, the net charge $\mathcal{Q}$ will be zero. There will, however, be a residual dipole moment.

This limit shows that the general solution (3.5) describes two oppositely charged, extremal magnetic black strings that are circular, i.e., black rings. Before discussing this configuration, let us recall what happens in the case of a single black ring. Such a ring carries a local distribution of charge that continues to be given by (3.8). However, if the S^2 is enlarged to enclose diametrically opposite points of the ring, the net charge $\mathcal{Q}$ will be zero. This is due to the opposite orientations of the individual S^2s enclosing the two points in five dimensions. Thus, the ring as a whole will have a zero net charge, and for this reason, it is also known as a dipole ring [19].*

Returning to the case of the two oppositely charged black rings, we see that there is already a local cancellation of charge when the S^2 encloses a point from each ring with the same S^1-coordinate ψ. We emphasize that this is even before the S^2 is extended to enclose the other two diametrically opposite points. When extended, the S^2 would enclose two dipoles with *opposite* dipole moments, resulting in a residual quadrupole moment.† In

*However, note that the context of the term 'dipole' is different from the one used in the preceding paragraph.

†This is in fact very similar to the situation when the accelerating dihole solution is extended past the acceleration horizon. There is another dihole pair lying in this new region of the space-time, mirroring the original dihole pair in the sense that its dipole moment is reversed. The extended space-time thus contains two dihole pairs oppositely oriented along a straight line, resulting in a quadrupole moment.

analogy with the case of black diholes, we shall refer to such an oppositely charged double-black-ring configuration as a ‘black di-ring’.

Another limit one can consider to confirm this di-ring interpretation, is to send the second black ring of Fig. 2 to infinity. One would then expect to recover the solution for a single static, extremal dipole ring, first constructed in [7, 8]. This is achieved by setting $A = 1/(a + \sqrt{m^2 + R^2})$, performing the coordinate transformation

$$x = 1 - \frac{(\sqrt{m^2+R^2}+m)(1-\tilde{x})}{a(\tilde{x}-\tilde{y})}\,, \qquad y = -1 + \frac{(\sqrt{m^2+R^2}-m)(1+\tilde{y})}{a(\tilde{x}-\tilde{y})}\,, \tag{3.9}$$

and then taking $a \to \infty$. The solution (3.5) reduces to, for an appropriate choice of K_0,

$$\begin{aligned} \mathrm{d}s^2 &= -\left(\frac{H(\tilde{x})}{H(\tilde{y})}\right)^{\frac{N}{3}} \mathrm{d}t^2 + \frac{R^2}{(\tilde{x}-\tilde{y})^2}\,\big(H(\tilde{x})H(\tilde{y})^2\big)^{\frac{N}{3}} \left[-\frac{1-\tilde{y}^2}{H(\tilde{y})^N}\,\mathrm{d}\psi^2 \right.\\ &\qquad \left. - \frac{\mathrm{d}\tilde{y}^2}{1-\tilde{y}^2} + \frac{\mathrm{d}\tilde{x}^2}{1-\tilde{x}^2} + \frac{1-\tilde{x}^2}{H(\tilde{x})^N}\,\mathrm{d}\varphi^2 \right], & \text{(3.10a)}\\ A_\varphi &= \sqrt{\frac{1+\mu}{1-\mu}}\,\frac{\sqrt{N}R\mu(1-\tilde{x})}{H(\tilde{x})}\,, & \text{(3.10b)}\\ \phi &= -\frac{\alpha N}{2}\ln\left(\frac{H(\tilde{x})}{H(\tilde{y})}\right), & \text{(3.10c)} \end{aligned}$$

where we have defined $H(\tilde{\xi}) = 1 - \mu\tilde{\xi}$ and $\mu = m/\sqrt{m^2+R^2}$. Indeed, this is precisely the solution for the dipole ring, written in the coordinates of [19].‡

We can also zoom in to each of the black rings directly. To zoom in to the first one, we perform the coordinate transformation (2.17b,c), and then take r to be smaller than any other length scale present. For an appropriately chosen K_0, the resulting metric is

$$\begin{aligned} \mathrm{d}s^2 &= g(\theta)^{\frac{N}{3}} \left[\left(\frac{r}{Q}\right)^{\frac{N}{3}} (-\mathrm{d}t^2 + r_1^2\mathrm{d}\psi^2) + \left(\frac{r}{Q}\right)^{-\frac{2N}{3}} (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) \right]\\ &\qquad + \left(\frac{r}{Q}\,g(\theta)\right)^{-\frac{2N}{3}} r^2\sin^2\theta\,\mathrm{d}\varphi^2, \end{aligned} \tag{3.11}$$

with the associated gauge field (2.18b) and dilaton (2.18c). Q and $g(\theta)$ continue to be given by (2.19). This solution describes the region near the $S^1 \times S^2$ horizon of an extremal dipole ring (with the horizon located at $r = 0$). It can be seen from the metric that the

‡Alternatively, one could consider shrinking the first black ring of Fig. 2 down to zero size, leaving just the second black ring. This limit also yields the solution for a single extremal dipole ring, and is in fact equivalent to taking the Myers–Perry black hole limit of the Figueras black ring [3].

S^1 ring radius is $r_1 \equiv [(1 - r_+A)(1 + r_-A)]^{\frac{1}{2}}/A$, while the S^2 sections of the horizon are distorted away from spherical symmetry by the presence of the factor $g(\theta)$. This distortion is expected from the self-attraction of the ring, as well as the mutual attraction of the other ring. Furthermore, it can be checked that this horizon is regular only in the non-dilatonic case $N = 3$ (although it has zero area). For the dilatonic rings, the horizons are actually null singularities.

Similarly, we may zoom in to the second ring with the transformation (2.20b,c). The resulting solution is given by the same solution as the above, but with the replacement $A \to -A$. In particular, the radius of this ring is $r_2 \equiv [(1+r_+A)(1-r_-A)]^{\frac{1}{2}}/A$; as expected, it is greater than that of the first ring. Otherwise, the above remarks for the first ring still apply.

The distortion of the two ring horizons is an indication that there are conical singularities attached to the rings, whose presence we shall now check for. If we take the angles φ and ψ to have the usual periodicity 2π, then the deficit angles along the three parts of the φ-axis are again given by (2.21), while the deficit angle along the ψ-axis is

$$\delta_{(y=-1)} = 2\pi \left[1 - (1 - 2mA - a^2A^2)^{\frac{N}{2}}\, K_0\right]. \tag{3.12}$$

It can be seen that setting $K_0 = (1 - 2mA - a^2A^2)^{-\frac{N}{2}}$ would simultaneously remove the conical singularities along the two semi-infinite axes $x = -1$ and $y = -1$. There will, however, be a resulting conical excess along $x = 1$, and also along $y = -\frac{1}{r_+A}$. The former is a disk spanning the interior of the first ring, whose presence is required to prevent the ring from collapsing under its own attraction. The latter is an annulus spanning the region between the two rings, keeping them apart in static equilibrium. Note that for this choice of K_0, the space-time is asymptotically flat, and its ADM mass is readily computed to be $\frac{\pi}{2G}\frac{Nm}{A}$.

Now, it is possible to immerse the di-ring in a background magnetic field, using a five-dimensional analogue of the Harrison transformation [8]. The resulting metric is

$$\begin{aligned} \mathrm{d}s^2 = &-\left(\frac{H(y,x)\Lambda}{H(x,y)}\right)^{\frac{N}{3}} \mathrm{d}t^2 + \frac{1}{A^2(x-y)^2}\,\big(H(y,x)H(x,y)^2\Lambda\big)^{\frac{N}{3}} \left[-\frac{(1-y^2)F(x)}{H(x,y)^N}\,\mathrm{d}\psi^2\right. \\ &\left. + \frac{1}{K_0^2\,G(x,y)^{N-1}}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} + \frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right) + \frac{(1-x^2)F(y)}{(H(y,x)\Lambda)^N}\,\mathrm{d}\varphi^2\right], \end{aligned} \tag{3.13}$$

with the associated gauge field and dilaton given by (2.23b) and (2.23c), respectively. In these expressions, Λ is defined as in (2.24), and B is a new parameter that determines the

strength of the background magnetic field. This solution would no longer be asymptotically flat, but would asymptote to a five-dimensional dilatonic analogue of the Melvin universe.

It can be checked that the background magnetic field will contribute to the distortion of the black-ring horizons. For the first black ring, the near-horizon metric is given by (3.11), with the distortion factor (2.25). For the second black ring, the metric is the same but we have to replace $A \to -A$ in the distortion factor. Furthermore, it can be verified that the presence of the magnetic field would only affect the conical singularity along $y = -\frac{1}{r_+ A}$, whose deficit angle is now given by (2.26). By appropriately adjusting B, it is possible to remove this conical singularity; in this case, there would no longer be any conical singularities attached to the second ring, and the S^2 sections of its horizon would become perfectly spherically symmetric. However, there will still be a conical excess along the inner disk $x = 1$, which cannot be removed by the magnetic field.

Finally, recall that in five dimensions, a solution that is magnetically charged with respect to the gauge potential A_a may be dualized into another solution that is electrically charged with respect to a two-form gauge potential B_{ab}. The duality transformation takes the form

$$\phi' = -\phi \,, \qquad H_{abc} = \frac{\mathrm{e}^{-\alpha\phi}}{2} \varepsilon_{abc}{}^{de} F_{de} \,, \tag{3.14}$$

where $H_{abc} \equiv \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$, and maps the action (3.3) to

$$\frac{1}{16\pi G} \int \mathrm{d}^5 x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi')^2 - \frac{1}{12} \mathrm{e}^{-\alpha\phi'} H^2 \right) . \tag{3.15}$$

Applying this duality transformation to the above solution would give one that describes an electrically charged di-ring immersed in a background electric field. The metric is still given by (3.13), while the dilaton is the negative of (2.23c). The two-form potential has a non-zero component $B_{t\psi}$, which is given by the same expression as the right-hand side of (2.28). The properties of this solution would be very similar to those of the magnetic solution. For the particular case of $N = 1$, it has the possible string-theory interpretation as two loops of fundamental string with opposite charges. More string-theory applications will be discussed in the conclusion.

4. Conclusion

The accelerating dihole solution that we have presented in this paper is, to the best of our knowledge, the first known example of a solution generalizing the charged C-metric

to more than one black hole accelerating in the same direction. (In the vacuum case, a solution describing multiple accelerating black holes has been constructed in [20].) While the existence of such a solution is perhaps not unexpected, what is notable is the relatively compact form in which it can be written. Unfortunately, this is unlikely to hold in any generalization of it. For example, the non-extremal dihole solution is known to be very complicated [13], and its accelerating version, if it can be found, would probably be even more so.

There are a variety of ways in which our solutions, particularly the di-ring, can be embedded in a higher-dimensional theory such as string or M-theory. The simplest way is to extend the world-volume of the di-ring by adding flat directions to it appropriately. One would then obtain concentric tubular brane–anti-brane pairs. For example, one can use this method to construct tubular D6–anti-D6-branes, with world-volume geometry $\mathbb{R}^{1,5} \times S^1$, in ten-dimensional Type IIA string theory. Another way is to regard the di-ring as an intersection of higher-dimensional branes. An M-theory realization of this consists of two sets of three M5-branes, with each set of M5-branes intersecting over a ring. All these solutions contain conical singularities, which are necessary to maintain equilibrium. Although it is possible to immerse them in a background magnetic field (also known as a fluxbrane in this context), this would not be sufficient to balance the forces present and remove all the conical singularities. For details on the construction of these solutions, we refer the reader to [8, 21].

Finally, we note that the rotating black ring solution of Figueras [3] can be turned into yet another solution of five-dimensional vacuum Einstein gravity. This solution can be read off from the results of Sec. 2, by taking the electrically charged accelerating dihole solution in Kaluza–Klein theory, and lifting it back to five dimensions with the appropriate analytic continuations. It can be checked that this new solution describes a pair of concentric, extremally rotating black rings. The rotation of these rings is now in the S^1 direction, as in the original rotating black ring of Emparan and Reall [1]. Moreover, they are rotating in directions *opposite* to each other. Because their rotation is extremal (in the sense that their corresponding rods have shrunk down to zero length), these rings are actually null singularities. Nevertheless, it should be interesting to study this solution in more detail.

A. Accelerating Zipoy–Voorhees solution

Of the black dihole class of solutions, the one belonging to pure Einstein–Maxwell theory is perhaps the most famous. It was first derived by Bonnor [22] in 1966, who recognized

that it represented a magnetic dipole source; for this reason, it is also known as the Bonnor dipole solution in the literature. In his paper, Bonnor noted that taking the coincident limit of this solution yields the neutral Darmois solution [23], which can be interpreted as two Schwarzschild black holes superposed on the top of each other in the Weyl formalism. The Darmois solution is, in fact, a special case of a more general class of static, axisymmetric vacuum solutions known as the Zipoy–Voorhees solution [10, 11]. The latter is parametrized by the real number δ, and can be interpreted as the superposition of δ Schwarzschild black holes when δ is a positive integer. The usual Schwarzschild solution is recovered when $\delta = 1$. For all other values of δ however, the space-times contain naked singularities, and so their physical interpretations remain unclear.

It therefore follows that the coincident limit of the accelerating Bonnor dipole solution is an accelerating version of the Darmois solution. It can be read off from (2.8a), by setting the dilaton coupling α and separation parameter a to zero. In fact, we can do better: it is possible to generalize this solution to include an arbitrary parameter δ, thereby obtaining an accelerating version of the Zipoy–Voorhees solution. The result is

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{A^2(x-y)^2}\bigg[&\left(\frac{F(y)}{F(x)}\right)^{\delta}(1-y^2)F(x)\,\mathrm{d}t^2 + \left(\frac{F(y)}{F(x)}\right)^{-\delta}(1-x^2)F(y)\,\mathrm{d}\varphi^2 \\ &+ \frac{(F(x)F(y))^{\delta(\delta-1)}}{G(x,y)^{\delta^2-1}}\left(-\frac{\mathrm{d}y^2}{(1-y^2)F(y)} + \frac{\mathrm{d}x^2}{(1-x^2)F(x)}\right)\bigg], \end{aligned} \tag{A.1}$$

where

$$F(\xi) = 1 + 2mA\xi\,, \qquad G(x,y) = (1+mA(x+y))^2 - m^2A^2(1-xy)^2. \tag{A.2}$$

Note that the usual C-metric (in the form written in [15]) is obtained when $\delta = 1$, while the accelerating Darmois solution is obtained when $\delta = 2$. If we perform the coordinate transformation (2.12) and take the zero acceleration limit $A \to 0$, we indeed recover the Zipoy–Voorhees solution from (A.1):

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{2m}{r}\right)^{\delta}\mathrm{d}t^2 + \frac{r^{\delta(\delta+1)}(r-2m)^{\delta(\delta-1)}}{[(r-m)^2 - m^2\cos^2\theta]^{\delta^2-1}}\left(\frac{\mathrm{d}r^2}{r(r-2m)} + \mathrm{d}\theta^2\right) \\ &+ \left(1-\frac{2m}{r}\right)^{1-\delta} r^2\sin^2\theta\,\mathrm{d}\varphi^2. \end{aligned} \tag{A.3}$$

It should be possible to analyze the space-time (A.1) in more detail using standard methods, but we will not do so here.

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